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STABILIZATION OF FREE ROTATION OF AN ASYMMETRIC TOP WITH CAVITIES COMPLETELY FILLED WITH A LIQUID

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We analyze the motion of an asymmetric top with cavities filled with a viscous incompressible liquid, and we study the stabilizing effect of the liquid on the rotation of the top around a given axis. The characteristic time for stabilization and the best orientation of the cavity relative to the solid body, have been found.

1. Equations of motion and their investigation. In a coordinate system whose axes are directed along the principal inertia axis of a body-liquid system, the equations of motion of a top for small Reynolds numbers reduce to the form [1]

$$I\omega^{\cdot} + [\omega, I\omega] = \frac{\rho}{\nu}\{P\omega^{\cdot\cdot} + [\omega, P\omega^{\cdot}]\} \tag{1.1}$$

Here I is the system's inertia tensor, ω is the top's angular velocity, ρ and ν are the liquid's density and kinematic viscosity, respectively. The right-hand side of (1.1) describes the force moment caused by the liquid's motion relative to the top; terms of a higher order of smallness with respect to ρ / ν are discarded. The tensor $P = \| P_{ij} \|$ is determined only by the shape of the cavity and is symmetric, $P_{ii} > 0$. In the case of several cavities this tensor equals the sum of the tensors for the individual cavities. The computation of the components of this tensor for a given cavity is a separate problem. It has been obtained in [1] for cavities of certain shapes. The motion of a solid with a symmetric cavity, when the tensor P is a multiple of the unit tensor, was studied in [1]. Here we examine the case of an arbitrary tensor P.

We rewrite Eq. (1.1) in the form

$$\mathbf{M}^{\cdot} + [\omega, \mathbf{M}] = 0, \qquad \mathbf{M} = I\omega - \frac{\rho}{\nu} P\omega^{\cdot}$$

where $\mathbf{M}$ is the system's total impulse moment. Hence right away we see the two relations

$$\mathbf{M}\mathbf{M}^{\cdot} = 0, \qquad \mathbf{M}^{\cdot}\omega = 0 \tag{1.2}$$

i.e. the law of preservation of the impulse moment and the law of dissipation of the system's energy

$$\Gamma^2 - 2\frac{\rho}{\nu}(\Gamma, P\omega^{\cdot}) = \text{const} \tag{1.3}$$

$$\frac{dE}{dt} = \frac{dH}{dt} - \frac{\rho}{\nu}\frac{d}{dt}(\omega, P\omega^{\cdot}) = -\frac{\rho}{\nu}(\omega^{\cdot}, P\omega^{\cdot}) < 0$$

$$E = \frac{1}{2}(\omega, I\omega) - \frac{\rho}{\nu}(\omega, P\omega^{\cdot})$$

Here E is the total energy of the system, $\Gamma = I\omega$ and $H = {}^1/_2\,(\omega, I\omega)$ are the impulse moment and the system's energy under the assumption that the liquid does not move relative to the solid. Since Eq. (1.1) is valid to within terms of order ρ^2 / ν^2, its subsequent investigation is carried out to the same accuracy. Accounting for the higher-order terms would require the determination of the following hydrodynamic approximation.

If we set $\nu = \infty$, the system's motion is an Euler-Poinsot motion (see [2], for example) which depends on three parameters: Γ^2, H and the time reference point. All these parameters are independent of time for infinite viscosity. For finite but large viscosity Eq. (1.3) contains a small parameter; therefore, we can take agvantage of the asymptotic methods developed in [3, 4]. The top's motion here is treated as the motion of a free top in which the parameters H and Γ vary slowly with time. Therefore, the period of the effective Euler-Poinsot motion also is a slowly varying function of time, and it is convenient to average relations (1.3) over the period of the unperturbed Euler-Poinsot motion. Disregarding the parameters' dependence on time during the averaging, we obtain

$$\Gamma^2 = \text{const}, \qquad \frac{dH}{dt} = -\frac{\rho}{\nu}(\omega^{\cdot}, P\omega^{\cdot}) < 0 \tag{1.4}$$

Here and subsequently, by Γ^2 and H we shall mean quantities averaged over the Euler-Poinsot period.

The hydrodynamic problem was solved on the assumption that ω, $\omega^{\cdot}$, and subsequent derivatives are of the order of unity. Therefore those solutions of Eqs. (1.1) and (1.3) which satisfy these conditions, are to be selected. The following solutions proposed in [1] for the approximate version of Eqs. (1.1):

$$I\omega^{\cdot} + [\omega, I\omega] = \frac{\rho}{\nu}\{P\mathbf{b} + [\omega, P\mathbf{a}]\} \tag{1.5}$$

$$\mathbf{a} = -I^{-1}[\omega, I\omega], \qquad \mathbf{b} = -I^{-1}[\mathbf{a}, I\omega] - I^{-1}[\omega, I\mathbf{a}]$$

where $\mathbf{a}$ and $\mathbf{b}$ are to within ρ^2 / ν^2 the same as $\omega^{\cdot}$ and $\omega^{\cdot\cdot}$, satisfy these conditions.

Let us investigate the stability of the system's stationary states, i.e. of the rotations around the principal inertia axes. Suppose that the motion is close to a rotation around the ith axis: $\omega = \Omega_i + \delta$. Then Eq. (1.5) takes the form

$$I\delta^{\cdot} - \frac{\rho}{\nu}P\mathbf{b}_\delta + [\Omega_i, I\delta] + I_i[\delta, \Omega_i] - \frac{\rho}{\nu}[\Omega_i, P\mathbf{a}_\delta] = 0$$

$$\mathbf{a}_\delta = -I^{-1}[\delta, I_i\Omega_i] - I^{-1}[\Omega_i, I\delta]$$

$$\mathbf{b}_\delta = -I^{-1}[\mathbf{a}_\delta, I_i\Omega_i] - I^{-1}[\Omega_i, I\mathbf{a}_\delta]$$

Setting $i = 1$, we obtain three first-order differential equations, two of which depend only on δ_2 and δ_3, while the third depends on all three δ_j. When developing the characteristic equation we should discard terms $\sim \rho^2 / \nu^2$. It turns out here that its roots depend only on the diagonal components of tensor $\| P_{ik} \|$ and are

$$\lambda_{1,2} = -\frac{\rho}{\nu}\frac{I_1}{2I_2I_3}\Omega_1^2\left(P_{22}\frac{I_1 - I_3}{I_2} + P_{33}\frac{I_1 - I_2}{I_3}\right) \pm i\Omega_1\sqrt{\frac{I_1 - I_2}{I_3}\frac{I_1 - I_3}{I_2}}$$

Hence it follows that if I_1 is not the largest moment of inertia, the steady-state rotations are unstable. Examining the case when I_1 is the maximum moment of inertia, we need to consider that there is one more root of the characteristic equation, $\lambda_3 = 0$.

Therefore, to investigate stability we set up the Liapunov function

$$V = 2I_1E - M^2 + (M^2 - I_1^2\Omega_1^2)^2$$

It equals zero for the steady-state motion, while in the case of small deviations from it, it equals the sum of the polynomial

$$V_1 = I_2(I_1 - I_2)\delta_2^2 + I_3(I_1 - I_3)\delta_3^2 + (I_1^2\delta_1^2 + I_2^2\delta_2^2 + I_3^2\delta_3^2 + 2I_1^2\delta_1\Omega_1)^2$$

and a polynomial proportional to ρ / ν (see [5], for example). When $I_1 > I_2, I_3$ the function V thus is a positive-definite function of δ_1, δ_2, δ_3. But from equalities (1.2) and (1.3) it follows that $V^\cdot \leqslant 0$. Thus, the rotation around the X_1-axis is stable by Liapunov's theorem, which agrees with the results in [1] and with the theorems in [5].

2. Averaged motion of a free top. Thus, for ω we shall investigate the Euler-Poinsot formulas in which energy is a slowly varying function of time. Equations (1.5) still remain sufficiently complicated; therefore, it is convenient to find H from Eq. (1.4). For definiteness we assume that $I_1 > I_2 > I_3$. We denote

$$a = \frac{I_1 - I_2}{I_3}, \quad b = \frac{I_2 - I_3}{I_1}, \quad c = \frac{I_1 - I_3}{I_2}$$

Let us examine the cases $2HI_2 < \Gamma^2$ (the trajectories Γ envelop the X_1-axis) and $2HI_2 > \Gamma^2$ (the trajectories Γ envelop the X_3-axis) separately. In the first case, as usual, instead of energy we introduce the dimensionless quantity

$$k_1^2 = \frac{2HI_1 - \Gamma^2}{\Gamma^2 - 2HI_3}\frac{I_1 b}{I_3 a}, \qquad 0 \leqslant k_1^2 \leqslant 1 \tag{2.1}$$

and instead of time, the dimensionless variable $\xi = (t - t_0) / T_0$, where T_0 is the characteristic time of the averaged motion. Substituting the Euler-Poinsot formulas into (1.4) and averaging over the period, we write this equation in the form:

$$\frac{dk_1^2}{d\xi} = (1 - \varkappa)(1 - k_1^2) - \frac{\mathrm{E}(k_1^2)}{\mathrm{K}(k_1^2)}[(1 - \varkappa) + k_1^2(1 + \varkappa)] \tag{2.2}$$

where $\mathrm{E}(k_1^2)$, $\mathrm{K}(k_1^2)$ are the complete elliptic integrals, and we have used the notation

$$\varkappa = 3\frac{P_{33}a - P_{11}b}{P_{33}a + 2P_{22}c + P_{11}b}, \qquad T_0 = \frac{\nu}{\rho}\frac{I_1I_2I_3}{\Gamma^2}\frac{3}{P_{33}a + 2P_{22}c + P_{11}b} \tag{2.3}$$

The terms containing the off-diagonal components of $\| P_{ik} \|$ fall out during the averaging. Thus, in the approximation being considered the top's motion depends only on the components of tensor $\| P_{ik} \|$ along the principal inertia axes, which agrees with the results obtained by computing the characteristic numbers for a motion close to the steady-state one.

Equation (2.2) coincides with the equation obtained in [1] for tensor P a multiple of the unit tensor, but here $\varkappa$ can take any value from the range $(-3, +3)$ instead of $[-1, +1]$. The quantity k_1^2 decreases monotonically from 1 to 0 as ξ increases from 0 to ∞. The solution of Eq. (2.2) can easily be found numerically. The corresponding graphs are shown in Fig. 1. Obviously, k_1^2 decreases with the growth of ξ the faster the larger $\varkappa$ is. For large values of ξ (small k_1^2) we can make use of the series expansions of E and K (see [6], for example). Restricting ourselves to quadratic terms, we obtain

$$\frac{dk_1^2}{d\xi} = -k_1^2\frac{3+\varkappa}{2}, \quad k_1^2 = \text{const}\exp\left(-\frac{3+\varkappa}{2}\xi\right) = C_1\exp\left(-\frac{t}{T_1}\right) \tag{2.4}$$

$$T_1 = \frac{\nu}{\rho}\frac{I_1I_2I_3}{\Gamma^2}\frac{1}{P_{22}c+P_{33}a} \tag{2.5}$$

For the case $2HI_2 > \Gamma^2$ it is convenient to introduce the quantity

$$k_3^2 = \frac{\Gamma^2-2HI_3}{2HI_1-\Gamma^2}\frac{I_3a}{I_1b} = \frac{1}{k_1^2}, \quad 0 \leqslant k_3^2 \leqslant 1 \tag{2.6}$$

The equation for k_3^2 coincides with Eq. (2.2), but the corresponding expressions for $\varkappa$ and T_0 differ in sign from (2.3). For small k_3^2 we obtain

$$k_3^2 = C_3\exp\frac{t}{T_3}, \quad T_3 = \frac{\nu}{\rho}\frac{I_1I_2I_3}{\Gamma^2}\frac{1}{P_{22}c+P_{11}b} \tag{2.7}$$

Note that as t increases k_3^2 grows towards unity.

Let us now examine how the system's averaged motion takes place in time. Suppose that at a certain instant the vector $\mathbf{\Gamma}$ is close to the X_3-axis. This signifies that $H \approx \Gamma^2 / 2I_3$. The motion is determined by formulas (2.6), (2.7) and $\mathbf{\Gamma}$ drifts away from the X_3-axis towards the X_2-axis, where k_3^2, $k_1^2 \approx 1$ $(H \approx \Gamma^2 / 2I_2)$. Subsequently H continues to decrease with time approaching the value $H = \Gamma^2 / 2I_1$. Here $\mathbf{\Gamma}$ approaches to the X_1-axis. Thus, only the motion around the X_1-axis is stable, in accord with the results in Sect. 1 and the general theorems in [5]. The motion of vector $\mathbf{\Gamma}$ coincides with that described in [1], differing only in the larger interval of possible values of $\varkappa$. It is obvious that the total characteristic time of recovery of orientation is of the order of $T_1 + T_3$ and thus, $\sim \nu$. Note also that $1 / T_0 = (1 / T_1) + (1 / T_3)$; consequently, $T_0 < T_1$, T_3.

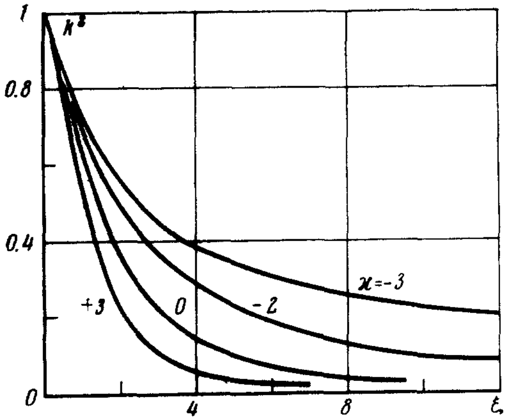

Fig. 1

It is clear that the most advantageous location of the cavity with liquid within the solid is determined by the requirement that both T_1 and T_3 be as small as possible. If the diagonal elements of P_{ik} are strongly different from each other and if the inertia tensor of the whole system depends weakly on the location of the cavity with liquid, then, as we see from (2.5) and (2.7), we need to orient the cavity relative to the solid so that the value of P_{22} turned out to be the largest of the eigenvalues of P_{ik}. This remark refers, in particular, to the case of a toroidal cavity, where one of the diagonal values of P_{ik} is considerably greater than the other two.

Generally speaking, the Euler-Poinsot period grows unboundedly as $k^2 \to 1$, and averaging becomes meaningless. But the solution is such only in a narrow range of values of energy when $\ln(1-k^2)^{-1} \gtrsim \nu$. It is not difficult to verify that for the averaged motion this segment is passed through in a time of the order of unity. This region corresponds to the location of Γ close to the X_2-axis, where even the usual Euler-Poinsot motion is unstable. Since the actual motion still contains oscillations, absent in the averaged motion, in reality this region is passed through more rapidly. Since the total orientation time $\sim \nu$, a motion close to the X_2-axis does not make an essential contribution in it. Close to the X_1-axis

$$\omega_1 = \frac{\Gamma}{I_1}\left(1 - \frac{1}{2}\frac{I_3^2 a}{I_1^2 b} k_1^2\right) \to \frac{\Gamma}{I_1}$$

$$\omega_2 = \frac{\Gamma}{I_1} k_1 \sqrt{\frac{c}{b}} \sin\left(t \frac{\Gamma}{I_1}\sqrt{ac}\right) \to 0$$

$$\omega_3 = \frac{\Gamma}{I_1} k_1 \sqrt{\frac{a}{b}} \cos\left(t \frac{\Gamma}{I_1}\sqrt{ac}\right) \to 0$$

$$\cos\theta = 1 - \frac{1}{2}\frac{I_3^2 a}{I_1^2 b} k_1^2 \to 1$$

where θ is the angle between Γ and the X_1-axis.

Let us briefly consider the case of a symmetric top. The dimensionless parameter corresponding to energy here proves to be simply an exponential function of time. The characteristic time of the process is

$$T_2 = \frac{\nu}{\rho}\frac{I_1 I_2^2}{a(P_{22}+P_{33})\Gamma^2} \quad \text{for} \quad I_1 > I_2 = I_3$$

$$T_4 = \frac{\nu}{\rho}\frac{I_1^2 I_3}{b(P_{22}+P_{11})\Gamma^2} \quad \text{for} \quad I_1 = I_2 > I_3$$

which coincides with the characteristic time of motion of the top around the X_1 or the X_3-axis, respectively. If $I_1 > I_2 = I_3$, then for any initial rotation the vector Γ goes to the X_1-axis along a spiral. However, in the case $I_1 = I_2 > I_3$ the vector Γ tends to some axis lying in the X_1X_2-plane. The position of this axis is determined by the initial conditions.

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